YITP-11-40

Asymptotic flatness at null infinity in arbitrary dimensions

Kentaro Tanabe, Shunichiro Kinoshita
Yukawa Institute for Theoretical Physics, Kyoto University, 606-8502 Kyoto, Japan

Tetsuya Shiromizu
Department of Physics, Kyoto University, 606-8502 Kyoto, Japan
(Dated: January 18, 2013)

We define the asymptotic flatness and discuss asymptotic symmetry at null infinity in arbitrary dimensions using the Bondi coordinates. To define the asymptotic flatness, we solve the Einstein equations and look at the asymptotic behavior of gravitational fields. Then we show the asymptotic symmetry and the Bondi mass loss law with the well-defined definition.

I. INTRODUCTION

Recently inspired by string theory, the systematic investigation of gravitational theory in higher dimensional spacetimes becomes more important. Indeed there are many things to be studied. Asymptotic structure of higher dimensional spacetimes is one of them. Asymptotically flat spacetimes have spatial and null infinities. While the asymptotic structure at spatial infinity has been investigated in arbitrary dimensions [1, 2], the studies on null infinity have been done only in even dimensions [3–9] and five dimensions [10, 11].

In four dimensions, asymptotic structure at null infinity was investigated in two ways. One is based on the Bondi coordinates [4, 5] and the another is based on the conformal embedding [3, 12]. In the latter, we introduce the conformal factor $\Omega \sim 1/r$ and we can study the behavior of gravitational fields near null infinity in the conformally transformed spacetime. This method can be extended to higher dimensions, but even dimensions only [7–9]. The reason why *even dimensions* is as follows. In n-dimensional spacetimes, gravitational fields behave like $\sim 1/r^{n/2-1} \sim \Omega^{n/2-1}$ and then we cannot suppose the smoothness of gravitational fields at null infinity due to the power of the half-integer in odd dimensions. Thus the conformal embedding method will not be useful for the investigation of the asymptotic structure at null infinity in any dimensions.

Instead of the conformal embedding method, we could safely define asymptotic flatness and study the asymptotic structure at null infinity in *five* dimensions by using the Bondi coordinates [10]. Therein one must solve the Einstein equations and determines the asymptotic behavior of gravitational fields which gives us a natural definition of asymptotic flatness at null infinity. We can show the asymptotic symmetry and the finiteness of the Bondi mass in five dimensions. The purpose of this paper is the extension of this work to arbitrary dimensions. For simplicity, we will consider the spacetimes satisfying the vacuum Einstein equation in higher dimensions. But, it is easy to extend our current work into nonvacuum cases that matters rapidly decay near null infinity.

The remaining part of this paper is organized as follows. In Sec. II, we introduce the Bondi coordinates in n-dimensional spacetimes and write down the Einstein equations in the language of the ADM formalism. In Sec. III, the Einstein equations will be solved explicitly and asymptotic flatness is defined by asymptotic behaviors at null infinity. We also define the Bondi mass and show its finiteness and the Bondi mass loss law. In Sec. IV, we shall study the asymptotic symmetry. In Sec. V we will summarize our work and discuss our future work. In Appendix A, we give the formulae of the ADM decomposition which will be used in Sec. III and in Appendix B we show the detail derivations of some equations.

II. BONDI COORDINATES AND ADM DECOMPOSITION

We introduce the Bondi coordinates in n-dimensional spacetimes. In the Bondi coordinates the metric can be written as

$$\mathrm{d}s^2 = -Ae^B \mathrm{d}u^2 - 2e^B \mathrm{d}u\mathrm{d}r + \gamma_{IJ}(\mathrm{d}x^I + C^I \mathrm{d}u)(\mathrm{d}x^J + C^J \mathrm{d}u), \tag{1}$$

where $x^a = (u, r, x^I)$ denote the retarded time, radial coordinate and angular coordinates, respectively. We also impose the gauge condition as

$$\sqrt{\det \gamma_{IJ}} = r^{n-2}\omega_{n-2}, \tag{2}$$

where ω_{n-2} is the volume element on the unit $(n-2)$-dimensional sphere. In the Bondi coordinates, null infinity is located at $r = \infty$.

We perform the ADM decomposition with respect to the r-constant surfaces (see Appendix A). The metric is rewritten as

$$ds^2 = N^2 dr^2 + q_{\mu\nu}(dx^\mu + N^\mu dr)(dx^\nu + N^\nu dr), \tag{3}$$

where

$$N^2 = \frac{e^B}{A}, \tag{4}$$
$$N^u = \frac{1}{A}, \tag{5}$$
$$N^I = -\frac{C^I}{A}, \tag{6}$$

and the induced metric of the r-constant surface

$$q_{\mu\nu} = \begin{pmatrix} -Ae^B + C^I C_I & C_J \\ C_I & \gamma_{IJ} \end{pmatrix}. \tag{7}$$

Note that the capital Latin indices $I, J, \cdots$ and the Greek indices $\mu, \nu, \cdots$ are raised and lowered using γ_{IJ} and $q_{\mu\nu}$, respectively. The unit normal vector to the r-constant surface is given by $n_a = N(dr)_a$ and $n^a = N^{-1}(\partial_r - N^\mu\partial_\mu)^a$. The extrinsic curvature of r-constant surface is defined as usual

$$K_{\mu\nu} = \frac{1}{2}\mathcal{L}_n q_{\mu\nu} = \frac{1}{2N}(\partial_r q_{\mu\nu} - \mathcal{D}_\mu N_\nu - \mathcal{D}_\nu N_\mu), \tag{8}$$

where $\mathcal{D}_\mu$ denotes the covariant derivative associated with $q_{\mu\nu}$.

The induced metric on the r-constant surface is rewritten as

$$q_{\mu\nu} dx^\mu dx^\nu = -\alpha^2 du^2 + \gamma_{IJ}(dx^I + \beta^I du)(dx^J + \beta^J du), \tag{9}$$

where

$$\alpha^2 = Ae^B, \quad \beta^I = C^I. \tag{10}$$

The timelike unit normal vector to the u-constant surface is written as $u_a = -\alpha(du)_a$ [1] and $u^a = \alpha^{-1}(\partial_u - \beta^I\partial_I)^a$. The extrinsic curvature of u-constant surface becomes

$$k_{IJ} = \frac{1}{2}\mathcal{L}_u \gamma_{IJ} = \frac{1}{2\alpha}(\partial_u \gamma_{IJ} - D_I\beta_J - D_J\beta_I), \tag{11}$$

where D_I denotes the covariant derivative associated with γ_{IJ}. We have $N^a = \alpha A^{-1}u^a = Nu^a$ from the definitions of u^a and N^a. Also we have $n^a = N^{-1}(\partial_r)^a - u^a$.

For later convenience we define the following projected quantities on the $(n-2)$-dimensional space as

$$\theta \equiv K_{\mu\nu}u^\mu u^\nu = -\tfrac{1}{N}\partial_r(\log\alpha) + \mathcal{L}_u \log N, \tag{12}$$
$$\rho^I \equiv K^I{}_\mu u^\mu = \tfrac{1}{2N\alpha}\partial_r\beta^I + \tfrac{1}{2}D^I \log\tfrac{N}{\alpha}, \tag{13}$$
$$\sigma_{IJ} \equiv K_{KL}\gamma_I{}^K\gamma_J{}^L = \tfrac{1}{2N}\partial_r\gamma_{IJ} - k_{IJ}, \tag{14}$$

where $\rho_\mu u^\mu = \sigma_{\mu\nu}u^\mu = 0$. Using of them, $K_{\mu\nu}$ is expressed by

$$K_{\mu\nu} = \theta u_\mu u_\nu - 2\rho_{(\mu}u_{\nu)} + \sigma_{\mu\nu}. \tag{15}$$

---

[1] This expression is valid on the induced manifold determined by the r-constant surface. For the whole spacetime manifold we should write it as $u_a = -\alpha(du)_a - N(dr)_a$

A. Decomposition of Ricci tensor

The vacuum Einstein equation is $\hat{R}_{ab}=0$. Let us decompose the n-dimensional Ricci tensor $\hat{R}_{ab}$ into the quantities on the $(n-2)$-dimensional space:

$$\begin{aligned}\hat{R}_{ab}n^a n^b =& \frac{1}{N}(\theta-\sigma)' - \mathcal{L}_u(\theta-\sigma) - \theta^2 + 2\rho^I\rho_I - \sigma_{IJ}\sigma^{IJ} + \frac{1}{N}\mathcal{L}_u\mathcal{L}_u N + k\mathcal{L}_u \log N \\ &- \frac{1}{N}D^2N - D_I\log\alpha D^I \log N,\end{aligned} \tag{16}$$

$$\hat{R}_{ab}n^a u^b = -\mathcal{L}_u\sigma - \theta k - k_{IJ}\sigma^{IJ} + D_I\rho^I + 2\rho^I D_I \log\alpha, \tag{17}$$

$$\begin{aligned}\hat{R}_{ab}u^a u^b =& -\frac{1}{N}\theta' + \theta^2 - \theta\sigma - 2\rho^I\rho_I + 2\rho^I D_I \log\frac{N}{\alpha} + D^I\log N D_I \log\alpha + \frac{1}{\alpha}D^2\alpha \\ &- \mathcal{L}_u k - k_{IJ}k^{IJ} + \mathcal{L}_u\theta - \frac{1}{N}\mathcal{L}_u\mathcal{L}_u N,\end{aligned} \tag{18}$$

$$\hat{R}_{ab}n^a\gamma^{bI} = \theta D^I\log\alpha - 2\rho^J k_J{}^I - \rho^I k + \sigma^{IJ}D_J\log\alpha + D_J\sigma^{IJ} - D^I\sigma + D^I\theta - \frac{1}{\alpha}[\dot{(\rho^I)} - \mathcal{L}_\beta\rho^I], \tag{19}$$

$$\begin{aligned}\hat{R}_{ab}u^a\gamma^{bI} =& D_J(k^{IJ} - \gamma^{IJ}k) - \sigma\rho^I - 2\rho_J\sigma^{IJ} - \frac{1}{N}D^I\mathcal{L}_u N + k^{IJ}D_J\log N \\ &- \frac{1}{N}(\rho^I)' + \theta D^I\log\frac{N}{\alpha} + \sigma^{IJ}D_J\log\frac{N}{\alpha} + \frac{1}{\alpha}[\dot{(\rho^I)} - \mathcal{L}_\beta\rho^I],\end{aligned} \tag{20}$$

$$\begin{aligned}\hat{R}_{ab}\gamma^{ab} =& -\frac{1}{N}\sigma' + \mathcal{L}_u\sigma + \theta\sigma - \sigma^2 - 2\rho^I\rho_I + 2\rho^I D_I\log\frac{N}{\alpha} \\ &+ \mathcal{L}_u k + k^2 - \frac{1}{\alpha}D^2\alpha - \frac{1}{N}D^2N + k\mathcal{L}_u\log N + {}^{(\gamma)}R\end{aligned} \tag{21}$$

and

$$\begin{aligned}\hat{R}_{ab}\gamma_I{}^a\gamma_J{}^b =& -\frac{1}{N}\sigma'_{IJ} + \frac{1}{\alpha}\dot{\sigma}_{IJ} - \frac{1}{\alpha}\mathcal{L}_\beta\sigma_{IJ} + \rho_I D_J\log\frac{N}{\alpha} + \rho_J D_I\log\frac{N}{\alpha} \\ &+ (\theta-\sigma)\sigma_{IJ} - 2\rho_I\rho_J + 2\sigma_{IK}\sigma_J{}^K - \frac{1}{N}D_ID_JN - \frac{1}{\alpha}D_ID_J\alpha \\ &+ \mathcal{L}_u k_{IJ} + kk_{IJ} - 2k_{IK}k_J{}^K + k_{IJ}\mathcal{L}_u\log N + {}^{(\gamma)}R_{IJ},\end{aligned} \tag{22}$$

where the prime and the dot respectively denote ∂_r and ∂_u, and ${}^{(\gamma)}R_{IJ}$ denotes the Ricci tensors with respect to γ_{IJ}. In the aboves, we have used the following equations

$$\begin{aligned}\gamma^{\mu\nu}\mathcal{L}_n K_{\mu\nu} =& \mathcal{L}_n\sigma + 2\sigma^{IJ}\sigma_{IJ} - 2\rho_I D^I\log\frac{N}{\alpha}, \\ u^\mu u^\nu\mathcal{L}_n K_{\mu\nu} =& \frac{1}{N}\theta' - \mathcal{L}_u\theta - 2\theta^2 + 4\rho^I\rho_I - 2\rho^I D_I\log\frac{N}{\alpha}, \\ q^{\mu\nu}\mathcal{L}_n K_{\mu\nu} =& \frac{1}{N}(-\theta+\sigma)' - \mathcal{L}_u(-\theta+\sigma) + 2\theta^2 - 4\rho^I\rho_I + 2\sigma_{IJ}\sigma^{IJ}, \\ \gamma_I{}^\mu\gamma_J{}^\nu\mathcal{L}_n K_{\mu\nu} =& \frac{1}{N}\sigma'_{IJ} - \frac{1}{\alpha}\dot{\sigma}_{IJ} + \frac{1}{\alpha}\mathcal{L}_\beta\sigma_{IJ} - \rho_I D_J\log\frac{N}{\alpha} - \rho_J D_I\log\frac{N}{\alpha}.\end{aligned} \tag{23}$$

III. ASYMPTOTIC FLATNESS AT NULL INFINITY AND BONDI MASS

In this section, we first solve the Einstein equations near null infinity and examine the asymptotic behaviors of the gravitational fields. Then these considerations give us the natural definition of the asymptotic flatness at null infinity. We also give the definition of the Bondi mass and momenta and then show its finiteness. In the following, we write γ_{IJ} as $\gamma_{IJ}=r^2h_{IJ}$ and the indices I,J are raised and lowered by h_{IJ}.

A. Constraint equations

Components of the vacuum Einstein equations $\hat{R}_{ra}=0$ and $\hat{R}_{ab}\gamma^{ab}=0$ are the constraint equations which do not contain u-derivative terms in the current coordinate system. Then, once we solve the equations $\hat{R}_{ra}=0$ on the initial u-constant surface, $\hat{R}_{ra}=0$ always hold in any u-constant surfaces.

After direct calculations $\hat{R}_{rr}=0$ becomes

$$B' = \frac{r}{4(n-2)} h'_{IJ} h'_{KL} h^{IK} h^{JL}, \tag{24}$$

where the prime stands for the r-derivative. From $\hat{R}_{ab}\gamma^{ab}=0$ we have

$$\begin{aligned}(n-2)\frac{(r^{n-3}A)'}{r^{n-2}} &= -\nabla_I C^{I\prime} - \frac{2(n-2)}{r}\nabla_I C^I - \frac{r^2 e^{-B}}{2} h_{IJ} C^{I\prime} C^{J\prime} \\ &\quad -\frac{e^B}{2r^2} h^{IJ}\nabla_I B \nabla_J B - \frac{e^B}{r^2}\nabla_I (h^{IJ}\nabla_J B) + \frac{e^B}{r^2}{}^{(h)}R,\end{aligned} \tag{25}$$

and from $\hat{R}_{rJ}\gamma^{IJ}=0$ we have

$$\frac{1}{r^{n-2}}(r^n e^{-B} h_{IJ} C^{J\prime})' = -\nabla_I B' + \frac{n-2}{r}\nabla_I B + {}^{(h)}\nabla^J h'_{IJ}, \tag{26}$$

where ∇_I and ${}^{(h)}\nabla_I$ denote the covariant derivative with respect to the metric of the unit $(n-2)$-sphere ω_{IJ} and h_{IJ}, respectively. ${}^{(h)}R$ is the Ricci scalar with respect to h_{IJ}.

Once h_{IJ} are given on the initial u-constant surface, the other metric functions A, B, C^I are automatically determined through the above equations on the initial u-constant surface. As seen later, it turns out that h_{IJ} contains the degree of freedom of gravitational waves in n-dimensional spacetimes.

We would suppose that h_{IJ} behaves near null infinity as follows

$$h_{IJ} \;=\; \omega_{IJ} + \sum_{k\geq 0} h^{(k+1)}_{IJ} r^{-(n/2+k-1)} = \omega_{IJ} + O(r^{-(n/2-1)}), \tag{27}$$

where the summation is taken over $k \in \mathbf{Z}$ for even dimensions and $2k \in \mathbf{Z}$ for odd dimensions. This comes from the fact that h_{IJ} corresponds to gravitational waves. By the gauge conditions of Eq. (2), $h^{(k+1)}_{IJ}$ should be traceless for $k < n/2-1$. Notice that we required the fall-off condition which is expected through the asymptotic behaviors of linear perturbations around the Minkowski spacetime. If the fall-off of h_{IJ} would be $O(r^k)$ for $k > -(n/2-1)$, the nonlinear feature would appear in the leading orders and then the Bondi mass will diverge. As shown later, the condition Eq. (27) corresponds to the outgoing boundary condition at null infinity.

By Eqs. (24) and (27), we can see that B behaves near null infinity as

$$B \;=\; B^{(1)} r^{-(n-2)} + O(r^{-(n-3/2)}), \tag{28}$$

where

$$B^{(1)} \;=\; -\frac{1}{16}\omega^{IK}\omega^{JL} h^{(1)}_{IJ} h^{(1)}_{KL}. \tag{29}$$

Substituting Eq. (26) into Eq. (27), we find that C^I should behave

$$C^I \;=\; \sum_{k=0}^{k<n/2-1} C^{(k+1)I} r^{-(n/2+k)} + J^I(u,x^I) r^{-(n-1)} + O(r^{-(n-1/2)}), \tag{30}$$

where

$$C^{(k+1)I} \;=\; \frac{2(n+2k-2)}{(n+2k)(n-2k-2)}\nabla_J h^{(k+1)IJ}. \tag{31}$$

$J^I(u,x^J)$ is the integration function in the r-integration. It corresponds to the angular momentum at null infinity. From the terms of the order of $O(r^{-(n-1)})$ terms in Eq. (26), we obtain the constraint conditions on $h^{(n/2)}_{IJ}$ as

$$\nabla^J h^{(n/2)}_{IJ} \;=\; 2\nabla_I B^{(1)}. \tag{32}$$

Substituting Eq. (25) into Eq. (27), we find that A should behave as

$$A = 1 + \sum_{k=0}^{k<n/2-2} A^{(k+1)} r^{-(n/2+k-1)} - m(u,x^I) r^{-(n-3)} + O(r^{-(n-5/2)}), \tag{33}$$

where

$$A^{(k+1)} = -\frac{2(n+2k-4)}{(n-2k-4)(n+2k-2)} \nabla^I C_I^{(k+1)} = -\frac{4(n+2k-4)}{(n+2k)(n-2k-2)(n-2k-4)} \nabla^I \nabla^J h_{IJ}^{(k+1)}. \tag{34}$$

$m(u,x^I)$ is the integration function in the r-integration. It corresponds to the energy and momentum at null infinity. From the terms of the order of $O(r^{-(n-1)})$ in Eq. (25), we obtain the constraint conditions on $h_{IJ}^{(n/2-1)}$ as

$$\nabla^I C_I^{(n/2-1)} = \nabla^I \nabla^J h_{IJ}^{(n/2-1)} = 0. \tag{35}$$

To be summarized, if we impose the boundary condition on h_{IJ} as Eq. (27), the behavior of other metric functions A, B and C^I near null infinity are determined. We will regard asymptotic behaviors as the definition of asymptotic flatness at null infinity in n-dimensional spacetimes as

$$h_{IJ} = \omega_{IJ} + O(r^{-(n/2-1)}),\ A = 1 + O(r^{-(n/2-1)}),\ B = O(r^{-(n-2)}),\ C^I = O(r^{-(n/2)}). \tag{36}$$

B. Bondi mass

Next we define the Bondi mass at null infinity in n-dimensional spacetimes. Since g_{uu} is expanded near null infinity as

$$g_{uu} = -1 - \sum_{k=0}^{k<n/2-2} \frac{A^{(k+1)}}{r^{n/2+k-1}} + \frac{m(u,x^I)}{r^{n-3}} + O(r^{-(n-5/2)}), \tag{37}$$

we define the Bondi mass $M_{\rm Bondi}(u)$ and momentum $M^i_{\rm Bondi}(u)$ as

$$M_{\rm Bondi}(u) \equiv \frac{n-2}{16\pi} \int_{S^{n-2}} m d\Omega, \tag{38}$$

$$M^i_{\rm Bondi}(u) \equiv \frac{n-2}{16\pi} \int_{S^{n-2}} m \hat{x}^i d\Omega, \tag{39}$$

respectively. $\hat{x}^i$ is the unit normal vector to the $(n-2)$-dimensional sphere which satisfies $\nabla_I \nabla_J \hat{x}^i + \omega_{IJ} \hat{x}^i = 0$. Thus each component of $\hat{x}^i$ is described by linear combination of the $l=1$ modes of the scalar harmonics on S^{n-2}. The Bondi energy-momentum is defined as $M^a_{\rm Bondi} = (M_{\rm Bondi}, M^i_{\rm Bondi})$.

In the conformal method [7–9], the Bondi mass is defined as $M_{\rm Bondi} \sim \int_{S^{n-2}} r^{n-1} \hat{C}_{urur} d\Omega \sim \int_{S^{n-2}} r^{n-1} \partial_r^2 g_{uu} d\Omega$, where $\hat{C}_{abcd}$ is the n-dimensional Weyl tensor. At first glance it seems to diverge at null infinity because $r^{n-1}\partial_r^2 g_{uu} \sim r^{n/2-2} A^{(1)}$ (it is shown that the Bondi mass is finite via an indirect argument in the conformal method [9].). However, since $A^{(k+1)}$ can be written as $A^{(k+1)} \propto \nabla^I \nabla^J h_{IJ}^{(k+1)}$ (see Eq. (34)), $A^{(k+1)}$ has no contribution to the mass and momentum at null infinity for $k < n/2-2$ as

$$\int_{S^{n-2}} A^{(k+1)} d\Omega = \int_{S^{n-2}} \hat{x}^i A^{(k+1)} d\Omega = 0. \tag{40}$$

Thus the Einstein equations guarantee the finiteness of the Bondi mass and momentum regardless of the dimension.

C. Evolution equations

The remaining components of the Einstein equation describe the evolution equations of gravitational fields. The equation $\hat{R}_{ab} \gamma^a{}_I \gamma^b{}_J = 0$ represents the evolutions of h_{IJ}. Indeed near null infinity we can obtain for $0 \le k < n/2-1$,

$$\begin{aligned}(k+1)\dot{h}_{IJ}^{(k+2)} &= -\frac{1}{2}(n-2k-4)A^{(k+1)}\omega_{IJ} + \frac{1}{8}[n^2-6n-(4k^2+4k-16)]h_{IJ}^{(k+1)} \\ &\quad + \frac{1}{2}(-\nabla^2 h_{IJ}^{(k+1)} + 2\nabla_{(I}\nabla^K h_{J)K}^{(k+1)}) - \frac{1}{2}(n-2k-4)\nabla_{(I}C_{J)}^{(k+1)} - \nabla^K C_K^{(k+1)}\omega_{IJ},\end{aligned} \tag{41}$$

where the dot denotes the u-derivative. Note that the evolutions of $h^{(1)}_{IJ}$ cannot be determined from the above equation. $\dot{h}^{(1)}_{IJ}$ are free functions on the initial u-constant surface. Contracting Eq. (41) with $\nabla^I\nabla^J$ and using Eq. (34), we can obtain the evolution equations of $A^{(k+1)}$ as

$$\dot{A}^{(k+2)} = -\frac{n+2k-2}{2(k+1)(n+2k+2)}\nabla^2 A^{(k+1)} + \frac{(n+2k-2)^2(n-2k-4)}{8(k+1)(n+2k+2)}A^{(k+1)}. \tag{42}$$

From $\hat{R}_{ab}u^a u^b = 0$, we can obtain the evolution equation of $m(u, x^I)$ as

$$\dot{m} = -\frac{1}{2(n-2)}\dot{h}^{(1)}_{IJ}\dot{h}^{(1)IJ} + \frac{n-5}{n-2}\nabla^I C^{(n/2-2)}_I + \frac{1}{n-2}\nabla^2 A^{(n/2-2)}. \tag{43}$$

Integrating this equation over the unit $(n-2)$-sphere, we can obtain the Bondi mass loss law as

$$\frac{d}{du}M_{\rm Bondi} = -\frac{1}{32\pi}\int_{S^{n-2}}\dot{h}^{(1)}_{IJ}\dot{h}^{(1)IJ}d\Omega \le 0. \tag{44}$$

Thus, the Bondi mass always decreases by gravitational waves and this justifies that our boundary conditions of Eq. (27) correspond to the outgoing boundary condition at null infinity.

IV. ASYMPTOTIC SYMMETRY

In this section we discuss the asymptotic symmetry at null infinity. We also confirm the Poincaré covariance of the Bondi mass and momentum.

A. Asymptotic symmetry

Asymptotic symmetry is defined to be the transformation group which preserves the asymptotic structure at null infinity. The variations of asymptotic form of metric at null infinity are given by

$$\delta g_{rr} = 0\,,\ \delta g_{rI} = 0\,,\ g^{IJ}\delta g_{IJ} = 0 \tag{45}$$

$$\delta g_{uu} = O(r^{-(n/2-1)})\,,\ \delta g_{uI} = O(r^{-(n/2-2)})\,,\ \delta g_{ur} = O(r^{-(n-2)})\,,\ \delta g_{IJ} = O(r^{-(n/2-3)}), \tag{46}$$

where $\delta g_{ab} \equiv \mathcal{L}_\xi g_{ab} = 2\hat{\nabla}_{(a}\xi_{b)}$ and ξ is the generator of asymptotic symmetry. In the following we consider the asymptotic symmetry in $n > 4$ dimensional spacetimes.

The condition of Eq. (45) comes from the definition of the Bondi coordinates and the explicit forms are

$$\delta g_{rr} = \mathcal{L}_\xi g_{rr} = -2e^B(\xi^u)' = 0, \tag{47}$$

$$\delta g_{rI} = \mathcal{L}_\xi g_{rI} = -e^B D_I\xi^u + \gamma_{IJ}C^J(\xi^u)' + \gamma_{IJ}(\xi^J)' = 0, \tag{48}$$

$$\gamma^{IJ}\delta g_{IJ} = \gamma^{IJ}\mathcal{L}_\xi g_{IJ} = \xi^r(\log\gamma)' + \xi^u(\dot{\log\gamma}) + 2D_I\xi^I + 2C^I D_I\xi^u = 0, \tag{49}$$

where $\gamma \equiv \det\gamma_{IJ}$. Then, using $\gamma_{IJ} = r^2 h_{IJ}$ and the gauge condition of Eq. (2), we can obtain ξ^a satisfying the above equations as

$$\xi^u = f(u, x^I), \tag{50}$$

$$\xi^I = f^I(u, x^I) + \int \mathrm{d}r\frac{e^B}{r^2}h^{IJ}\nabla_J f(u, x^I), \tag{51}$$

$$\xi^r = -\frac{r}{n-2}(C^I\nabla_I f + \nabla_I\xi^I). \tag{52}$$

For later convenience, we write down the asymptotic behavior of ξ near null infinity as

$$\xi_r \;=\; -f(u,x^I)+O(r^{-(n-2)}), \tag{53}$$

$$\begin{aligned}\xi_I \;&= r^2\omega_{IJ}f^J(u,x^K)-r\nabla_I f(u,x^K)\\ &+\sum_{k=0}^{k<n/2-2}\left(rh^{(k+1)}_{IJ}f^J+fC^{(k+1)}_I-\frac{n+2k-2}{n+2k}h^{(k+1)}{}_I{}^J\nabla_J f\right)r^{-(n/2+k-2)}+O(r^{-(n-3)}),\end{aligned} \tag{54}$$

$$\begin{aligned}\xi_u \;=\; \frac{r}{n-2}\Bigg[\nabla_I f^I-\frac{1}{r}(\nabla^2 f+(n-2)f)+ \quad (n-2)\sum_{k=0}^{k<n/2-2}\Bigg(&f^I C^{(k+1)}_I-\frac{fA^{(k+1)}}{r}-\frac{n+2k-4}{n+2k-2}\frac{C^{(k+1)}_I\nabla^I f}{r}\\ &+\frac{2}{n+2k}\frac{h^{(k+1)}_{IJ}\nabla^I\nabla^J f}{(n-2)r}\Bigg)r^{-(n/2+k-1)}\Bigg]+O(r^{-(n-3)}).\end{aligned} \tag{55}$$

Next let us consider the boundary conditions of Eq. (46). The each components of metric variations are

$$\delta g_{uu} \;=\; \frac{2r}{n-2}\frac{\partial}{\partial u}\nabla_I f^I-\frac{2}{n-2}\frac{\partial}{\partial u}(\nabla^2 f+(n-2)f)+2C^{(1)}_I\partial_u f^I r^{-(n/2-2)}+O(r^{-(n/2-1)}), \tag{56}$$

$$\delta g_{ur} \;=\; \frac{1}{n-2}[\nabla_I f^I-(n-2)\partial_u f]-\sum_{k=0}^{k<n/2-2}\frac{n+2k-2}{(n-2)(n+2k)}h^{(k+1)}_{IJ}\nabla^I\nabla^J f r^{-(n/2+k)}+O(r^{-(n-2)}), \tag{57}$$

$$\begin{aligned}\delta g_{uI} \;&=\; r^2\partial_u f_I+\frac{r}{n-2}\partial_I[\nabla_J f^J-(n-2)\partial_u f]-\frac{1}{n-2}\partial_I[\nabla^2 f+(n-2)f]\\ &+h^{(1)}_{IJ}\partial_u f^J r^{-(n/2-3)}+O(r^{-(n/2-2)}),\end{aligned} \tag{58}$$

$$\delta g_{IJ} \;=\; 2r^2\left[\nabla_{(I}f_{J)}-\frac{\nabla_K f^K}{n-2}\omega_{IJ}\right]-2r\left[\nabla_I\nabla_J f-\frac{\nabla^2 f}{n-2}\omega_{IJ}\right]+O(r^{-(n/2-3)}). \tag{59}$$

To satisfy the boundary conditions of Eq. (46) for these equations, we will find that f and f^I should satisfy

$$\partial_u f^I \;=\; 0, \tag{60}$$

$$\nabla_I f_J+\nabla_J f_I \;=\; \frac{2\nabla_K f^K}{n-2}\omega_{IJ},\quad \nabla_I f^I=(n-2)\frac{\partial f}{\partial u}, \tag{61}$$

$$\nabla_I\nabla_J f \;=\; \frac{\nabla^2 f}{n-2}\omega_{IJ}. \tag{62}$$

Integrating the trace part of Eq. (61), we can obtain

$$f \;=\; \frac{F(x^I)}{n-2}u+\alpha(x^I), \tag{63}$$

where $F\equiv\nabla_I f^I$ and $\alpha(x^I)$ is an integration function on S^{n-2}. Here we can show from Eqs. (62) and (63) that F satisfies

$$\nabla_I\nabla_J F \;=\; \frac{1}{n-2}\omega_{IJ}\nabla^2 F, \tag{64}$$

and also contracting Eq. (61) with $\nabla^I\nabla^J$ we have

$$\nabla^2 F+(n-2)F \;=\; 0. \tag{65}$$

The general solutions to these equations for F are the $l=1$ modes of the scalar harmonics on S^{n-2}. Next from Eqs. (62) and (63) we can see that

$$\nabla_I\nabla_J\alpha \;=\; \frac{1}{n-2}\omega_{IJ}\nabla^2\alpha. \tag{66}$$

should hold in $n > 4$ dimensions. The general solutions to this equations are $l = 0$ and $l = 1$ modes of the scalar harmonics on S^{n-2}.

To be summarized, f can be written as

$$f \,=\, f_0 + f_1(u, x^I), \tag{67}$$

where f_0 is a constant and corresponds to the $l = 0$ mode in α. $f_1(u, x^I)$ contains the $l = 1$ modes in F and α for $n > 4$ dimensions. Thus we can show that f satisfies the following equations:

$$\nabla_I(\nabla^2 f + (n-2)f) \,=\, 0\,,\, \partial_u(\nabla^2 f + (n-2)f) \,=\, 0, \tag{68}$$

in $n > 4$ dimensions. In addition, since $\nabla_I \nabla_J f \propto \omega_{IJ}$ and $h^{(k+1)}_{IJ}$ are traceless for $k < n/2 - 1$, the gauge condition of Eq. (2) implies that $h^{(k+1)}_{IJ} \nabla^I \nabla^J f$ vanishes for $k < n/2 - 2$ in Eq. (57). As a consequence, we could confirm that the transformations satisfying Eqs. (60), (61) and (62) keep the boundary conditions (46).

It is worth noting that Eq. (61) gives another condition for $f^{(\mathrm{tra})I}$ which is the transverse part of f^I, namely satisfying $\nabla_I f^{(\mathrm{tra})I} = 0$. We find that Eq. (61) corresponds to the Killing equation $\nabla_I f^{(\mathrm{tra})}_J + \nabla_J f^{(\mathrm{tra})}_I = 0$ on S^{n-2} because of the transverse condition. This means that $f^{(\mathrm{tra})I}$ is the Killing vector on S^{n-2}. Therefore, the transformations generated by the transverse part of f^I are trivial and we could focus on only the longitudinal part of f^I which generates nontrivial transformations.

Here we give the short summary. We could show that the asymptotic symmetry is generated by f and f^I satisfying Eqs. (60), (61) and (62). The parts of f, which are not proportional to u, generates a translation group. f^I generates the Lorentz group. Then the asymptotic symmetry at null infinity is the Poincaré group.

Before closing this subsection, we have a comment on four dimensional cases for the comparison. In four dimensional cases, the boundary conditions to be held are

$$\delta g_{uu} \,=\, O(r^{-1})\,,\, \delta g_{ur} \,=\, O(r^{-2})\,,\, \delta g_{uI} \,=\, O(1)\,,\, \delta g_{IJ} \,=\, O(r). \tag{69}$$

Then, if f and f^I satisfy Eqs. (60) and (61), the transformations keep the above boundary conditions. Note that the condition (62) is not required because the second term in the right-hand side in Eq. (59) already satisfies the boundary conditions and has no additional restriction to f in four dimensions. Therefore there is no restriction on α where $f = F(x^I)u/2 + \alpha$. Hence α is an arbitrary functions on S^2 in four dimensions while in $n > 4$ dimensions α should be $l = 0$ or $l = 1$ mode. The former condition $\nabla_I(\nabla^2 f + (n-2)f) \,=\, 0$ in Eq. (68), which comes from Eq. (58), does not hold in four dimension because α can have $l > 1$ modes. However since the third term in the right-hand side in Eq. (58) already satisfies the boundary conditions (69) in four dimensions, that condition is not required. The $l > 1$ modes of α correspond to the generators of the so-called supertranslation group. Thus the asymptotic symmetry is the semi-direct group of supertranslation and Lorentz group in four dimensions rather than the Poincaré group.

B. Poincaré covariance

Next we shall confirm the Poincaré covariance of the Bondi mass and momentum. Since the asymptotic symmetry is the Poincaré group, we expected that the Bondi mass and momentum should be transformed covariantly under the action of its Poincaré group. In practice, under the translation of $f = \alpha(x^I)$ and $f^I = 0$, the Bondi energy-momentum is invariant, that is,

$$M^a_{\mathrm{Bondi}} \to M^a_{\mathrm{Bondi}}. \tag{70}$$

However, since we consider dynamical spacetimes, it is easy to expect the contribution from gravitational waves under translation $u \to u - \alpha$. Thus the Bondi energy-momentum M^a_{Bondi} should be transformed under the translation as

$$M^a_{\mathrm{Bondi}}(u) \to M^a_{\mathrm{Bondi}}(u) + \mathcal{L}_\xi M^a_{\mathrm{Bondi}} = M^a_{\mathrm{Bondi}}(u) + \alpha \frac{d}{du} M^a_{\mathrm{Bondi}}(u), \tag{71}$$

where the second term in the right-hand side represents the effect of gravitational radiations. Let us look at the details.

For the translations, the generator ξ_a becomes

$$\xi_r \,=\, -\alpha + O(r^{-(n-2)}), \tag{72}$$

$$\xi_I = -r\nabla_I\alpha + \sum_{k=0}^{k<n/2-2}\left[\alpha C_I^{(k+1)} - \frac{n+2k-2}{n+2k}h_{IJ}^{(k+1)}\nabla^J\alpha\right]r^{-(n/2+k-2)} + O(r^{-(n-3)}), \tag{73}$$

$$\xi_u = -\sum_{k=0}^{k<n/2-2}\left[\frac{n+2k-4}{n+2k-2}C_I^{(k+1)}\nabla^I\alpha + \alpha A^{(k+1)}\right]r^{-(n/2+k-1)} + O(r^{-(n-3)}). \tag{74}$$

δg_{uu} can be computed as

$$\begin{aligned}\delta g_{uu} &= 2\hat{\nabla}_u\xi_u \\ &= \sum_{k=0}^{k=n/2-2}\delta g_{uu}^{(k+1)}r^{-(n/2+k-1)} + O(r^{-(n-5/2)}),\end{aligned} \tag{75}$$

where

$$\delta g_{uu}^{(k+1)} = \frac{4}{n+2k-2}\partial_u C_I^{(k+1)}\nabla^I\alpha - \alpha\partial_u A^{(k+1)} - \frac{n+2k-4}{2}\alpha A^{(k)} + \nabla^I\alpha\nabla_I A^{(k)}. \tag{76}$$

In particular, for $k = n/2-2$

$$\begin{aligned}\delta g_{uu}^{(n/2-1)} &= \delta m \\ &= \alpha\partial_u m + \frac{2}{n-3}\nabla^I\alpha\partial_u C_I^{(n/2-1)} - \alpha(n-4)A^{(n/2-2)} + \nabla^I\alpha\nabla_I A^{(n/2-2)}.\end{aligned} \tag{77}$$

By using Eqs. (34), (42) and (35), we can rewrite $\delta g_{uu}^{(k+1)}$ as

$$\begin{aligned}\delta g_{uu}^{(k+1)} = \frac{2}{n+2k}[\nabla^2(\alpha A^{(k)}) + (n-2)\alpha A^{(k)}] \;+\; &\frac{4}{(n+2k)(n-2k-2)}\nabla^I\nabla^J(\alpha\partial_u h_{IJ}^{(k+1)}) \\ - \;&\frac{2(n+2k-6)}{(n+2k)(n-2k-2)}[\nabla^I\nabla^J(\nabla_I\alpha C_J^{(k)}) + C_I^{(k)}\nabla^I\alpha],\end{aligned} \tag{78}$$

for $0 \le k < n/2-2$ and

$$\begin{aligned}\delta m &= \alpha\partial_u m + \frac{2}{n-3}\nabla^I\alpha\partial_u C_I^{(n/2-1)} - (n-4)\alpha A^{(n/2-2)} + \nabla^I\alpha\nabla_I A^{(n/2-2)} \\ &= -\frac{\alpha}{2(n-2)}\dot{h}_{IJ}^{(1)}\dot{h}^{(1)IJ} + \frac{2}{n-2}\nabla^I\nabla^J(\alpha\partial_u h_{IJ}^{(n/2-1)}) + \frac{1}{n-2}[\nabla^2(\alpha A^{(n/2-2)}) + (n-2)\alpha A^{(n/2-2)}] \\ &\qquad - \frac{n-5}{n-2}[\nabla^I\nabla^J(\nabla_I\alpha C_J^{(n/2-2)}) + C_I^{(n/2-2)}\nabla^I\alpha],\end{aligned} \tag{79}$$

for $k = n/2-2$. We can show that

$$\int_{S^{n-2}}\delta g^{(k+1)}d\Omega = 0\,, \quad \text{and} \quad \int_{S^{n-2}}\hat{x}^i\delta g_{uu}^{(k+1)}d\Omega = 0, \tag{80}$$

for $k < n/2-2$ and

$$\int_{S^{n-2}}\delta m d\Omega = -\frac{1}{2(n-2)}\int_{S^{n-2}}\alpha\dot{h}_{IJ}^{(1)}\dot{h}^{(1)IJ}d\Omega, \tag{81}$$

for $k = n/2-2$. See Appendix B for the details of the calculations.

Equation (80) implies that translations $u \to u - \alpha$ preserve the finiteness of the Bondi energy-momentum. Equation (81) can be rewritten as

$$M_{\mathrm{Bondi}}^a \to M_{\mathrm{Bondi}}^a + \alpha\frac{d}{du}M_{\mathrm{Bondi}}^a, \tag{82}$$

where dM_{Bondi}/du is given by Eq. (44). Thus the Bondi mass in our definition has the Poincaré covariance under the asymptotic symmetry.

V. SUMMARY AND OUTLOOK

In this paper, we have investigated the asymptotic structure at null infinity in n-dimensional spacetimes using the Bondi coordinates. Asymptotic flatness is defined by the asymptotic behavior of gravitational fields at null infinity. These boundary conditions are determined by solving the Einstein equations. Although the Bondi mass seems to diverge in the conformal method, we can show its finiteness from the Einstein equations in the Bondi coordinates. And we can show that asymptotic symmetry at null infinity should be the Poincaré group and the Bondi energy-momentum is transformed covariantly under the Poincaré group by using the Einstein equations. These results are same with those in [7, 8] for even dimensions. Note that the conditions for asymptotic flatness in [7, 8] come from the stability of weak asymptotic simplicity [13] . On the other hands, our definition of asymptotic flatness comes from the behavior of perturbations around the Minkowski spacetime. In general, these two definitions may differ. The Bondi mass will diverge unless our boundary conditions at null infinity are not satisfied. In this sense we would expect that our definition guarantees the stability of weak asymptotic simplicity at null infinity. Nevertherless, it is nice to show that our definition is generic enough regardless of the Minkowski spacetime as Refs. [7, 8].

As our future work we will be able to consider angular momentum at null infinity in n-dimensional spacetimes. Since asymptotic symmetry at null infinity is the Poincaré group without supertranslations in higher dimensions, we can define the angular momentum. Indeed, we can define the angular momentum and show its Poincaré covariance in five dimensions [11].

Acknowledgments

KT is supported by JSPS Grant-Aid for Scientific Research (No. 21-2105). SK is the Yukawa Fellow and this work is partially supported by Yukawa Memorial Foundation. TS is partially supported by Grant-Aid for Scientific Research from Ministry of Education, Science, Sports and Culture of Japan (Nos. 21244033, 21111006, 20540258 and 19GS0219). This work is also supported by the Grant-in-Aid for the Global COE Program “The Next Generation of Physics, Spun from Universality and Emergence” from the Ministry of Education, Culture, Sports, Science and Technology (MEXT) of Japan.

Appendix A: $((n-1)+1)$-decomposition

The n-dimensional metric can be written as

$$g_{ab} = \epsilon n_a n_b + \gamma_{ab}, \tag{A1}$$

where γ_{ab} is $(n-1)$-dimensional induced metric and n^a is the unit normal vector, which is normalized by $n_a n^a = \epsilon$. Note that ϵ takes $+1$ or -1 which means the normal vector is spacelike or timelike, respectively.

We define the extrinsic curvature as

$$K_{ab} = \frac{1}{2}\mathcal{L}_n \gamma_{ab}. \tag{A2}$$

Because n_a is the normal vector to the $(n-1)$-dimensional hypersurface, it can be written as $n_a = \epsilon N \nabla_a \Omega$ where Ω is a function which describes the hypersurface by $\Omega = \text{const.}$ and N is so-called lapse function. Then, the Riemann tensor becomes

$$R_{efgh}\gamma_a{}^e\gamma_b{}^f\gamma_c{}^g\gamma_d{}^h = {}^{(\gamma)}R_{abcd} - \epsilon K_{ac}K_{bd} + \epsilon K_{ad}K_{bc}, \tag{A3}$$

$$R_{efgd}\gamma_a{}^e\gamma_b{}^f\gamma_c{}^g n^d = D_a K_{bc} - D_b K_{ac}, \tag{A4}$$

$$R_{acbd}n^c n^d = -\mathcal{L}_n K_{ab} + K_{ac}K_b{}^c - \epsilon\frac{1}{N}D_a D_b N, \tag{A5}$$

where D_a denotes the covariant derivative with respect to γ_{ab}. Note that we have used $n^a\nabla_a n_b = -\epsilon D_b \log N$.

The Ricci tensor becomes

$$R_{ab}n^a n^b = -\mathcal{L}_n K - K_{ab}K^{ab} - \epsilon\frac{1}{N}D^2 N, \tag{A6}$$

$$R_{ac}n^a\gamma_b{}^c = D^a K_{ab} - D_b K, \tag{A7}$$

$$R_{cd}\gamma_a{}^c\gamma_b{}^d = {}^{(\gamma)}R_{ab} - \epsilon\mathcal{L}_n K_{ab} - \epsilon K K_{ab} + 2\epsilon K_{ac}K_b{}^c - \frac{1}{N}D_a D_b N \tag{A8}$$

The Ricci scalar becomes

$$\begin{aligned} R &= {}^{(\gamma)}R - 2\epsilon\mathcal{L}_n K - \epsilon K^2 - \epsilon K_{ab}K^{ab} - \frac{2}{N}D^2 N \\ &= {}^{(\gamma)}R + \epsilon K^2 - \epsilon K_{ab}K^{ab} - \frac{2}{N}D^2 N - 2\epsilon\nabla_a(K n^a) \end{aligned} \tag{A9}$$

The each components of the Einstein tensor are given by

$$G_{ab}n^a n^b = \frac{1}{2}(-\epsilon^{(\gamma)}R + K^2 - K_{ab}K^{ab}), \tag{A10}$$

$$G_{ac}n^a\gamma_b{}^c = D^a K_{ab} - D_b K, \tag{A11}$$

$$\begin{aligned} G_{cd}\gamma_a{}^c\gamma_b{}^d = {}& {}^{(\gamma)}G_{ab} - \epsilon K K_{ab} + 2\epsilon K_{ac}K_b{}^c + \frac{\epsilon}{2}\gamma_{ab}(K_{cd}K^{cd} + K^2) \\ & - \epsilon\mathcal{L}_n K_{ab} + \epsilon\gamma_{ab}\mathcal{L}_n K - \frac{1}{N}D_a D_b N + \frac{1}{N}\gamma_{ab}D^2 N \end{aligned} \tag{A12}$$

Appendix B: Derivations of (80) and (81).

We will show Eq. (80) and (81). At first we show the former equation in Eq. (80). Since the integrations of the total derivative terms vanish, we can obtain

$$\int_{S^{n-2}} \delta g_{uu}^{(k+1)} d\Omega \;=\; \int_{S^{n-2}} \left[\frac{2(n-2)}{n+2k}\alpha A^{(k)} - \frac{2(n+2k-6)}{(n+2k)(n-2k-2)}\nabla^I \alpha C_I^{(k)}\right] d\Omega. \tag{B1}$$

Using Eqs. (31) and (34), we can see that

$$\begin{aligned} \int_{S^{n-2}} C^{(k)I}\nabla_I\,\alpha\, d\Omega \;&=\; \frac{2(n+2k-4)}{(n+2k-2)(n-2k)}\int_{S^{n-2}} \nabla_J h^{(k)IJ}\nabla_I \alpha d\Omega \\ &=\; -\frac{2(n+2k-4)}{(n+2k-2)(n-2k)}\int_{S^{n-2}} h^{(k)IJ}\nabla_I\nabla_J\alpha d\Omega \\ &=\; 0, \end{aligned} \tag{B2}$$

and

$$\begin{aligned} \int_{S^{n-2}} \alpha A^{(k)} d\Omega \;&=\; -\frac{4(n+2k-6)}{(n+2k-2)(n-2k)(n-2k-2)}\int_{S^{n-2}} \alpha\nabla^I\nabla^J h_{IJ}^{(k)} d\Omega \\ &=\; -\frac{4(n+2k-6)}{(n+2k-2)(n-2k)(n-2k-2)}\int_{S^{n-2}} h_{IJ}^{(k)}\nabla^I\nabla^J\alpha d\Omega \\ &=\; 0, \end{aligned} \tag{B3}$$

where we used the fact that $h_{IJ}^{(k)}$ are traceless for $k < n/2$. Then we can show

$$\int_{S^{n-2}} \delta g_{uu}^{(k+1)} d\Omega \;=\; 0. \tag{B4}$$

This is the former one in Eq. (80).

Next we show the latter one in Eq. (80). For this we can see

$$\int_{S^{n-2}} \hat{x}^i[\nabla^2(\alpha A^{(k)}) + (n-2)\alpha A^{(k)}] d\Omega \;=\; \int_{S^{n-2}} \alpha A^{(k)}[\nabla^2\hat{x}^i + (n-2)\hat{x}^i] d\Omega \;=\; 0, \tag{B5}$$

$$\int_{S^{n-2}} \hat{x}^i \nabla_I \nabla_J (\alpha h_{IJ}^{(k)}) d\Omega \;=\; \int_{S^{n-2}} \alpha h_{IJ}^{(k)} \nabla^I \nabla^J \hat{x}^i d\Omega \;=\; 0, \tag{B6}$$

and

$$\int_{S^{n-2}} \hat{x}^i [\nabla^I \nabla^J (\nabla_I \alpha C_J^{(k)}) + \nabla^I \alpha C_I^{(k)}] d\Omega \;=\; \int_{S^{n-2}} \nabla_I \alpha C_J^{(k)} [\nabla^I \nabla^J \hat{x}^i + \omega^{IJ} \hat{x}^i] d\Omega \;=\; 0 \tag{B7}$$

hold. In the aboves we used the tracelessness of $h_{IJ}^{(k)}$. Using of them, we can show

$$\int_{S^{n-2}} \hat{x}^i \delta g_{uu}^{(k+1)} d\Omega \;=\; 0. \tag{B8}$$

This is the latter one in Eq. (80).

Finally we show Eq. (81). Since the integrations on S^{n-2} of the total derivative terms vanish,

$$\begin{aligned} \int_{S^{n-2}} \delta m d\Omega \;&=\; -\frac{1}{2(n-2)} \int_{S^{n-2}} \alpha \dot{h}_{IJ}^{(1)} \dot{h}^{(1)IJ} d\Omega + \int_{S^{n-2}} \left[\alpha A^{(n/2-2)} - \frac{n-5}{n-2} C^{(n/2-2)I} \nabla_I \alpha \right] d\Omega \\ &=\; -\frac{1}{2(n-2)} \int_{S^{n-2}} \alpha \dot{h}_{IJ}^{(1)} \dot{h}^{(1)IJ} d\Omega, \end{aligned} \tag{B9}$$

where we used Eqs. (B2) and (B3) from the first to second line. This is Eq. (81).

---

[1] A. Ashtekar and R. O. Hansen, J. Math. Phys. **19**, 1542 (1978).
[2] K. Tanabe, N. Tanahashi and T. Shiromizu, J. Math. Phys. **50**, 072502 (2009) [arXiv:0902.1583 [gr-qc]].
[3] R. P. Geroch, "Asymptotic structure of space-time," in *Asymptotic structure of space-time* (edited by T.P. Esposito and L. Witten,(New York, Plenum), 1977).
[4] H. Bondi, M. G. J. van der Burg and A. W. K. Metzner, Proc. Roy. Soc. Lond. A **269**, 21 (1962).
[5] R. K. Sachs, Proc. Roy. Soc. Lond. A **270**, 103 (1962).
[6] R. K. Sachs, Phys. Rev. **128**, 2864 (1962).
[7] S. Hollands and A. Ishibashi, J. Math. Phys. **46**, 022503 (2005)
[8] S. Hollands and A. Ishibashi, arXiv:hep-th/0311178.
[9] A. Ishibashi, Class. Quant. Grav. **25**, 165004 (2008) [arXiv:0712.4348 [gr-qc]].
[10] K. Tanabe, N. Tanahashi and T. Shiromizu, J. Math. Phys. **51**, 062502 (2010)
[11] K. Tanabe, N. Tanahashi and T. Shiromizu, J. Math. Phys. **52**, 032501 (2011) arXiv:1010.1664 [gr-qc].
[12] R. Penrose, Phys. Rev. Lett. **10**, 66 (1963).
[13] R. P. Geroch and B. C. Xanthopoulos, J. Math. Phys. **19**, 714 (1978).